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Group Theory and Kink Stability Equations

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Starting from Schrödinger equations with $SU(2)$ group-theoretic potentials, we consider a general family of kinks labeled by two (half-)integers (l, n) with $|n| \leq l$. A particular choice of $n=0$, $l=L$ (L positive integer) leads to a general L-family, where $L=1$ corresponds to sine-Gordon theory, while $L=2$ represents the $(\lambda\phi^4)_{1+1}$ model. The $(\lambda\phi^6)_{1+1}$ model can also be recovered with $l=3/2$, $n=-1/2$, a particular case of theories labeled by l and n such that $l-n=2$ which possess simple kink solutions. We also discuss one-loop order corrections to the kink masses in supersymmetric versions of the L-family. As a byproduct, we obtain the SUSY renormalization of the so-called γ parameter in sine-Gordon theory.

Nonlinear field theories in $(1+1)$ dimensions, with Lagrangian density $\mathscr{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V[\phi]$, for which the classical equations of motion exhibit time-independent but space-dependent solutions of finite energy, are widely in use. We recall the so-called kink solution in the $(\lambda\phi^4)_{1+1}$ model as well as the more restrictive case of solitons in sine-Gordon theory. We can take small quantum perturbations over the classical solutions so that a linearization limit of the general field equations provides the so-called stability equations. We can also take the inverse of this procedure: Starting from the stability equations, we can define general nonlinear field theories. In particular, this program can be carried out for a general family of models labeled by a positive integer L, where $L=1$ reproduces the sine-Gordon theory and $L=2$ gives the $(\lambda\phi^4)_{1+1}$ model (Boya and Casahorran, 1989).

In this paper we discuss how we can obtain stability equations in the first place. We first notice that the L-family of stability equations is nothing but the time-independent Schrödinger equation with Pöschl and Teller (1933) potential (up to a constant shift in the potential). This potential is a well-known soluble example. In view of recent work relating soluble

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Schrödinger equations with group theory (e.g., Gürsey, 1983), the Pöschl–Teller potential is derived from the Casimir invariant of $SU(2)$. We thus can obtain candidates of stability equations as soluble Schrödinger equations derived from group theory. We thus propose a general method of generating nonlinear models from stability equations obtained from an arbitrary group.

In the second part of the paper we consider the supersymmetric version of the L-family. In particular, we compute the one-loop correction to the kinks' classical masses using a unified treatment for both bosonic and fermionic contributions. As a byproduct we obtain the supersymmetry renormalization of the γ parameter in the sine-Gordon system.

Let us now consider the stability equation for the L-family for positive integer L (Boya and Casahorran, 1989). Splitting the field as $\phi=\phi_c+\varphi$, where ϕ_c is the classical kink solution and φ the perturbation, we get the stability equation

$$\left[-\frac{d^2}{dx^2}+m^2L^2-\frac{m^2L(L+1)}{\cosh^2 mx}\right]\varphi_n(x)=\omega^2_{Bn}\varphi_n(x) \tag{1}$$

with discrete spectrum $\omega^2_{Bn}=m^2[L^2-(L-n)^2]$, $n=0, 1, \ldots, L-1$. We point out that this is nothing but a one-dimensional time-independent Schrödinger equation with an attractive Pöschl and Teller (1933) potential,

$$\left[-\frac{d^2}{dx^2}-\frac{j(j+1)}{\cosh^2 x}\right]\psi^m_j(x)=-m^2\psi^m_j(x) \tag{2}$$

with solutions $\psi^m_j(x)=P^m_j(\tanh x)$, where $P^m_j(z)$ is an associated Legendre polynomial. Note that here we can have j half-integers as well as integers. It is well known that the $j=$ integer case gives no reflection, whereas the $j=$ half-integer case gives maximum reflection (Morse and Feshbach, 1953). We also remark that the zero-mode solution of equation (1) is the highest weight solution of equation (2), namely $P^j_j(\tanh x)$. These observations let us to conjecture the following ansatz: For each Schrödinger equation associated with group-theoretic potential, there is a family of nonlinear models with kink solutions whose stability equations correspond to the given Schrödinger equation. Furthermore, the zero modes of the stability equations are obtained from the highest weight solution of the Schrödinger equation.

In light of this we can understand the kink stability equation of the $(\lambda\phi^6)_{1+1}$ model (Lohe, 1979) as an $SU(2)$ potential problem. In this model, there is the static solution

$$\phi_c(x)=\left[\frac{\mu}{2\lambda}(\tanh \mu x+1)\right]^{1/2} \tag{3}$$

of mass $M=\mu^2/4\lambda$, for the potential

$$V[\phi]=\tfrac{1}{2}\lambda^2\phi^2\left(\phi^2-\frac{\mu}{\lambda}\right)^2, \qquad \mu, \lambda>0 \tag{4}$$

The stability equation is the Schrödinger equation with potential

$$V''[\phi_c(x)]=\frac{5\mu^2}{2}+\frac{3\mu^2}{2}\tanh \mu x-\frac{15\mu^2}{4\cosh^2 \mu x} \tag{5}$$

This potential problem has been solved (Morse and Feshbach, 1953). To reveal the $SU(2)$ nature of the potential, let us consider the polynomial $P^l_{mn}(z)$ (Vilenkin, 1968). It is related to the Jacobi polynomials $P_k^{(\alpha,\beta)}(z)$ and the associated Legendre polynomials $P_l^m(z)$ as follows:

$$P_k^{(\alpha,\beta)}(z)=2^m i^{n-m}\left[\frac{(l-n)!\,(l+n)!}{(l-m)!\,(l+m)!}\right]^{1/2}(1-z)^{(n-m)/2}(1+z)^{(-n-m)/2}P^l_{mn}(z) \tag{6}$$

where

$$l=k+\frac{\alpha+\beta}{2}, \qquad m=\frac{\alpha+\beta}{2}, \qquad n=\frac{\beta-\alpha}{2} \tag{7}$$

and

$$P_l^m(z)=i^m\left[\frac{(l+m)!}{(l-m)!}\right]^{1/2}P^l_{m0}(z), \qquad m\geq 0 \tag{8}$$

The differential equation for $P^l_{mn}(\tanh x)$ is

$$\left[-\frac{d^2}{dx^2}-\frac{l(l+1)}{\cosh^2 x}-2mn\tanh x\right]P^l_{mn}(\tanh x)=-(m^2+n^2)P^l_{mn}(\tanh x) \tag{9}$$

We now see that we obtain the zero mode of the stability equation for the $(\lambda\phi^6)_{1+1}$ model for the choice of parameters $l=3/2$, $m=3/2$, and $n=-1/2$. We thus observe that if we regard l and n as parameters, we get the zero-mode solution of the stability equation with maximum allowed value of m. So the zero mode will be proportional to

$$P^l_{ln}(z)=\frac{i^{l-n}}{2^l}\left[\frac{(2l)!}{(l-n)!\,(l+n)!}\right]^{1/2}(1-z)^{(l-n)/2}(1+z)^{(l+n)/2} \tag{10}$$

In particular, for the $(\lambda\phi^6)_{1+1}$ kink, the zero mode is given by

$$\frac{d\phi_c}{dx} \sim (1-\tanh \mu x)(1+\tanh \mu x)^{1/2} \tag{11}$$

which agrees with the derivative of equation (4). We point out, at least for the group $SU(2)$, the existence of nonlinear field theories with stable kinks and stability equations with the group-theoretic potential.

Now let us discuss the general (l, n) case. The zero mode is given by

$$\frac{d\phi_c}{dx} \sim (1-\tanh x)^{(l-n)/2}(1+\tanh x)^{(l+n)/2} \tag{12}$$

so we have the classical kink solution as

$$\phi_c(y)-\phi_c(y_0) \sim \int_{y_0}^{y} dx(1-\tanh x)^{(l-n)/2}(1+\tanh x)^{(l+n)/2} \tag{13}$$

and the integration can be done with partial integration. As in Boya and Casahorran (1989) the supersymmetric quantum mechanical formulation of the stability equation relates the zero-mode solution to the superpotential $W(x)$ as follows:

$$\exp\left[-\int W(x)\, dx\right] = \frac{d\phi_c}{dx} \tag{14}$$

where $W(x) = U'[\phi_c(x)]$, for $\frac{1}{2}U[\phi\,]^2 = V[\phi\,]$. Of particular interest are the theories with $l-n=2$. For these cases we can have very simple kink solutions, labeled by l,

$$\phi_c(x) = (1+\tanh x)^{l-1} \tag{15}$$

The potential for the theories is obtained using equation (14) as follows:

$$V[\phi\,] = \tfrac{1}{2}(l-1)^2\phi^2(\phi^{1/(l-1)}-2)^2 \tag{16}$$

where we have adjusted the integration constant so as to make the minimum of the potential to be zero. We see that $l=3/2$ gives the ϕ^6 potential and $l=2$ the ϕ^4 potential. For larger values of l we have fractionally powered potentials, which are pathological, even though they do have well-defined kink solutions.

Returning to the general L-family, we can consider the supersymmetric version of the model which appears in Boya and Casahorran (1989). In particular, the quantum corrections to the classical kink masses have been widely discussed by many authors (Schonfeld, 1979; Kaul and Rajaraman, 1983). As a matter of fact, the kink mass receives nonzero quantum corrections, although the Witten–Olive bound remains saturated (Imbimbo and

Mukhi, 1984). Our next goal will be the first quantum correction to the L-dependent kink masses using a unified formula which only considers discrete levels of Schrödinger equations. By reason of the supersymmetric properties, we finally find a simple expression that depends upon the zero-energy mode. In particular, we can make good use of the result associated to the $L=1$ case (sine-Gordon) in order to obtain the SUSY renormalization of the γ parameter appearing in the S-matrix treatments (Zamolodchikov and Zamolodchikov, 1979). As a matter of fact, the actions we are looking for represent particular cases of a SUSY σ-model (Imbimbo and Mukhi, 1984). On elimination of the so-called "auxiliary field" we can write

$$S=\frac{1}{2}\int[(\partial_\mu\phi)^2+\bar{\psi}(i\gamma^\mu\partial_\mu)\psi-U_L^2[\phi]-U_L'[\phi]\bar{\psi}\psi]\,d^2x \tag{17}$$

where ϕ is a real scalar field, while ψ represents a Majorana field. The U_L^2 term corresponds to the L-family scalar potential and the prime denotes a derivative with respect to the argument. In principle, we recall the bosonic stability equation over the kink, namely equation (1). Going to the fermionic fields, it suffices to write the spinor in its two-component form,

$$\psi(x,t)=\begin{bmatrix}v_+(x)\\ v_-(x)\end{bmatrix}\exp(i\omega_F t) \tag{18}$$

to discover the hidden supersymmetric quantum mechanical character of the Dirac equation over the background provided by the scalar classical solution. In our case we will have

$$\left[-\frac{d^2}{dx^2}+m^2L^2-\frac{m^2L(L+1)}{\cosh^2 mx}\right]v_a(x)=\omega_F^2 v_a(x),\qquad a=+\text{ or }- \tag{19a}$$

$$\left[-\frac{d^2}{dx^2}+m^2L^2-\frac{m^2L(L-1)}{\cosh^2 mx}\right]v_a(x)=\omega_F^2 v_a(x),\qquad a=-\text{ or }+ \tag{19b}$$

where the explicit $a=+$ or $-$ identification in (19) depends on the Bogomolny condition sign. We can point out the discrete spectrum of (19b), namely $\omega_{Fn}^2=m^2[L^2-(L-n)^2]$, $n=1,\ldots,L-1$. In order to concoct a simple formula which provides us the first quantum correction to the classical kink mass in the SUSY case we only need the relation between the bosonic (fermionic) fluctuation modes in the continuum spectrum. Calling n_F the fermionic density of states per wave number in the continuum, it can easily be found that (Imbimbo and Mukhi, 1984)

$$n_F=\frac{1}{2}(n_++n_-) \tag{20}$$

where n_+ and n_- represent the densities associated with the Schrödinger operators which appear in equation (19). With these data at hand we can write the SUSY version of the simple formula of Boya and Casahorran (1989). By reason of the supersymmetric properties of the model (the 1/2 factor for the fermionic fluctuations and the duplication phenomena for the nonzero-energy eigenstates) we finally find that

$$\Delta M = -\frac{mL}{2\pi} \tag{21}$$

Taking the $L=1$ case, we obtain the correction associated with the sine-Gordon soliton, a result obtained in Lee *et al.* (1986) following the tedious phase shift and stability angles procedures. We recall the SUSY sine-Gordon theory, a model governed by the following Lagrangian density (Imbimbo and Mukhi, 1984):

$$\mathscr{L} = \frac{1}{2}\left\{(\partial_\mu\phi)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu)\psi - \frac{2m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}\,\phi}{m}\right)\right] - m\cos\left(\frac{\sqrt{\lambda}\,\phi}{2m}\right)\bar{\psi}\psi\right\} \tag{22}$$

Since SUSY does not modify the classical soliton mass, we can write at one-loop order that

$$M_{\text{SUSY}} = \frac{8m}{\tilde{\gamma}} \tag{23}$$

where

$$\tilde{\gamma} = \frac{\lambda/m^2}{1-\lambda/16\pi m^2} \tag{24}$$

so that the supersymmetric effect modifies the γ parameter with respect to the conventional sine-Gordon model result. The same phenomenon maintains its validity going to the SUSY breather or doublet solutions. Starting from the explicit classical solution (Dashen *et al.*, 1975) written in terms of period τ and dimensionless parameter $\tilde{\tau} = m\tau/2\pi$,

$$\phi_\tau(x,t) = \frac{4m}{\sqrt{\lambda}}\tan^{-1}\left((\tilde{\tau}^2-1)^{1/2}\frac{\sin(mt/\tilde{\tau})}{\cosh[mx(\tilde{\tau}^2-1)^{1/2}/\tilde{\tau}]}\right) \tag{25}$$

the action per period would be

$$S_{cl}[\phi_\tau] = \frac{32\pi m^2}{\lambda}\left[\cosh^{-1}\left(\frac{1}{\tilde{\tau}}\right) - (\tilde{\tau}^2-1)^{1/2}\right] \tag{26}$$

while the "one-loop" effects transform equation (26) into

$$S_q[\phi_\tau] = \frac{32\pi m^2}{\tilde{\gamma}} \left[\cos^{-1}\left(\frac{1}{\tilde{\tau}}\right) - (\tilde{\tau}^2 - 1)^{1/2} \right] \tag{27}$$

Resorting to the time-dependent WKB method, we can obtain the bound-state energy levels, namely

$$M_N = \frac{16m}{\tilde{\gamma}} \sin\left(\frac{N\tilde{\gamma}}{16}\right), \qquad N = 1, 2, \ldots < \frac{8\pi}{\tilde{\gamma}} \tag{28}$$

In this paper we initiated a program of generating a family of nonlinear models with kink solutions whose stability equations obey Schrödinger equations with group-theoretic potentials. We have also considered supersymmetric versions of theories associated with $SU(2)$. The next logical step would be to perform an analysis of the SUSY system using supergroups. Next, we know that the sine-Gordon equation is closely related to $SU(2)$ affine Lie algebra, and that its soliton stability equation was an $SU(2)$ potential problem. There might be a relation between these two. If there is such a relation, it would be an interesting task to establish the connection between general Toda field theory associated with affine G Lie algebra and nonlinear soliton equations obtained from a stability equation with potential associated with group G. Note that these theories intrinsically involve more than single scalar fields (Rajaraman, 1979). The first step would be to consider the stability equation derived from the Casimir operator of $SU(3)$, which might give nonlinear theories involving two scalar fields. This work is the subject of our next investigation.

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